09.07.2018

<u>Recall</u>: $E \xrightarrow{p} B$ is a fibre bundle (loc. trivial fibration) with fibre F if $\forall b \in B\ \exists U \ni b$ open s.t.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p|_{p^{-1}U}"'] & \circlearrowright & U \times F \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

$$\begin{tikzcd} C \times_B E = f^*E \arrow[r, "pr_E"] \arrow[d, "pr_C"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ C \arrow[r, "f"] & B \end{tikzcd}$$

pullback, $C \times_B E = \{(c,e) \in C \times E \mid f(c) = p(e)\}$

Univ. prop.: if X is a space with morphisms $X \xrightarrow{\alpha} E$, $X \xrightarrow{\beta} C$ s.t. $p\alpha = f\beta$ then $\exists!\ X \xrightarrow{\varphi} C \times_B E$ s.t. $pr_E \circ \varphi = \alpha$, $pr_C \circ \varphi = \beta$

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\varphi"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\ & f^*E \arrow[r, "pr_E"] \arrow[d, "pr_C"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ & C \arrow[r, "f"] & B \end{tikzcd}$$

If $p: E \longrightarrow B$ is a fibre bundle with fibre F then $C \times_B E \xrightarrow{pr_C} C$ is also a fibre bundle with fibre F (shown last week):

$\forall c \in C\ \exists U$: $f(c) \in U$ and

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & \circlearrowright & U \times F \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

$\Rightarrow c \in f^{-1}(U)$,

$$\begin{tikzcd} pr_C^{-1}(f^{-1}(c)) \arrow[rr, "\cong"] \arrow[dr, "pr_C"'] & \circlearrowright & f^{-1}(U) \times F \arrow[dl, "pr_{f^{-1}(U)}"] \\ & f^{-1}(U) & \end{tikzcd}$$

We will use this to show that every fibre bundle is a Serre fibration.

<u>Lemma</u>. If $p: E \longrightarrow I^n$ is a fibre bundle with fibre F then p is trivial

i.e.

$$\begin{tikzcd} E \arrow[rr, "\cong"] \arrow[dr, "p"'] & \circlearrowright & I^n \times F \arrow[dl, "pr_{I^n}"] \\ & I^n & \end{tikzcd}$$

Pf: Notation: $I_1^n := \{(t_1, \ldots, t_n) \in I^n \mid 0 \leq t_n \leq \frac{1}{2}\}$

$I_2^n := \{(t_1, \ldots, t_n) \in I^n \mid \frac{1}{2} \leq t_n \leq 1\}$

Claim. Let $E_1 := p^{-1}(I_1^n)$, $E_2 := p^{-1}(I_2^n)$. If $p|_{E_1}$ and $p|_{E_2}$ are trivial then $p: E \longrightarrow B$ is trivial.

$$\begin{tikzcd} E_1 \arrow[rr, shift left, "\varphi"] \arrow[dr, "p|_{E_1}"'] \arrow[rr, phantom, "\cong"] & & I_1^n \times F \arrow[ll, shift left, "\varphi^{-1}"] \arrow[dl, "pr_{I_1^n}"] \\ & I_1^n \arrow[u, phantom, "G"] & \end{tikzcd} \qquad \begin{tikzcd} E_2 \arrow[rr, shift left, "\psi"] \arrow[dr, "p|_{E_2}"'] \arrow[rr, phantom, "\cong"] & & I_2^n \times F \arrow[ll, shift left, "\psi^{-1}"] \arrow[dl, "pr_{I_2^n}"] \\ & I_2^n \arrow[u, phantom, "G"] & \end{tikzcd}$$

$$\begin{tikzcd} (I_1^n \cap I_2^n) \times F \arrow[r, shift left, "\psi^{-1}"] \arrow[dr, "pr_{I_1^n \cap I_2^n}"'] & E_1 \cap E_2 \arrow[l, shift left, "\psi"] \arrow[r, shift left, "\varphi"] \arrow[d, "p"] & (I_1^n \cap I_2^n) \times F \arrow[l, shift left, "\varphi^{-1}"] \arrow[dl, "pr_{I_1^n \cap I_2^n}"] \\ & I_1^n \cap I_2^n & \end{tikzcd}$$

clutching data

$$\left.\begin{array}{ll} \varphi\psi^{-1}(t,x) = (t, f(t,x)) & \text{for some } f: (I_1^n \cap I_2^n) \times F \longrightarrow F \\ \psi\varphi^{-1}(t,x) = (t, g(t,x)) & \text{for some } g: (I_1^n \cap I_2^n) \times F \longrightarrow F \end{array}\right\} \text{clutching functions}$$

These f and g are the obstructions to gluing the two trivialisations together.

Let $\pi: I_2^n \longrightarrow I_1^n \cap I_2^n$ be the projection to the "middle hyperplane"

$$(t_1, \dots, t_n) \longmapsto (t_1, \dots, t_{n-1}, \tfrac{1}{2})$$

This gives us maps $\begin{tikzcd} I_2^n \times F \arrow[r, shift left, "\lambda"] & I_1^n \times F \arrow[l, shift left, "\mu"] \end{tikzcd}$

$$(t,x) \longmapsto f(\pi(t), x)$$

$$(t, g(\pi(t)), x) \longleftarrow (t,x)$$

and λ, μ are mutually inverse to each other. (Checking this is not hard, one just has to use how f and g are def'd.)

Thus we obtain

$$\begin{tikzcd} I_2^n \times F \arrow[rr, shift left, "\psi^{-1}\mu"] \arrow[dr, "pr_{I_2^n}"'] & & E_2 \arrow[ll, shift left, "\lambda\psi"] \arrow[dl, "p|_{E_2}"] \\ & I_2^n \arrow[u, phantom, "G"] & \end{tikzcd} \qquad \begin{tikzcd} I_1^n \times F \arrow[rr, shift left, "\varphi^{-1}"] \arrow[dr] & & E_1 \arrow[ll, shift left, "\varphi"] \arrow[dl] \\ & I_1^n & \end{tikzcd}$$

compatibility on $I_1^n \cap I_2^n$.

Let $e \in E_1 \cap E_2$, thus $\psi(e) \in (I_1^n \times I_2^n) \times F$ since ψ is compat w/ projections.

WTC: $\lambda\psi(e) = \varphi(e)$

$$\psi(e) = (t,x) \in I_2^n \times F \qquad \forall t \in I_1^n \times I_2^n$$

$$\lambda\psi(e) = \lambda(t,x) = (t, f(\pi(t),x)) = (t, f(t,x)) = \varphi\psi^{-1}(t,x) = \varphi(e) \checkmark$$

Similarly: on $(I_1^n \cap I_2^n) \times F$ one has $\varphi^{-1} = \psi^{-1}\mu$.

Thus we have established compatibility on $I_1^n \cap I_2^n$. Actually, this is not even necessary: this follows from the above and the fact that these are homeos.

Define $E \xrightarrow{\alpha} I^n \times F$ by $\alpha|_{E_1} = \varphi$, $\alpha|_{E_2} = \psi$. By above discussion α is a well-defined. homeo

Similarly one has $E \underset{\beta}{\overset{\alpha}{\rightleftarrows}} I^n \times F$ ($\cong$). Thus proves the Claim. $\boxed{C}$

Generally, given $p: E \longrightarrow I^n$ fibre bundle:

cover I^n by $U_1, \dots, U_\ell$ where $p|U_i$ is trivial $\forall i$, i.e.

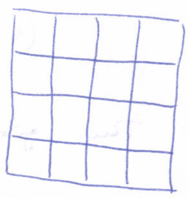
$$\begin{tikzcd} p^{-1}U_i \arrow[rr, "\cong"] \arrow[dr] & & U_i \times F \arrow[dl] \\ & U_i & \end{tikzcd}$$

Use Lebesgue number of this cover: let this be $\varepsilon > 0$,

i.e. $\forall A \subseteq I^n$ with $\mathrm{diam}(A) < \varepsilon$: $A \subseteq U_i$ for some i.

Subdivide I^n into small cubes with diameters $< \varepsilon$.

Then p is trivial on every small cube because of the property of ε and the triviality of $p|U_i$.



Use the Claim & induction $\Rightarrow$ the assertion follows. $\square$

Prop. Let $E \xrightarrow{p} B$ be a fibre bundle with fibre F. Then p is a Serre fibration.

Pf:

$$\begin{tikzcd} t \arrow[d, mapsto] & \in & I^n \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ (t,0) & \in & I^n \times I \arrow[r, "F"] \arrow[ur, dashed, "\exists ?"] & B \end{tikzcd}$$

We want to find a lift, i.e. the dashed arrow.

$$
\begin{tikzcd}
I^n \arrow[drrr, bend left, "f"] \arrow[dr, dashed, "\tilde{f}"] \arrow[dd, hook] & & & \\
& I^n \times I \times_B E \arrow[rr, "pr_E"] \arrow[d, "q"] & & E \arrow[d, "p"] \\
I^n \times I \arrow[r, equal] \arrow[ur, dashed, "\exists \alpha"] \arrow[urrr, "pr_E \circ \alpha"'] & I^n \times I \arrow[rr] & & B
\end{tikzcd}
$$

Then q is a fibre bundle, $p \circ f = F \circ \mathrm{incl}$

By the previous Lemma: q is trivial. (q is homeomorphic to the projection of a Cartesian product) $\Rightarrow$ q is a Serre fibration. □

Applications

$$S^1 \longrightarrow S^3 \xrightarrow{\mu} S^2 \qquad \text{where } S^2 = \text{Riemann sphere},$$
$$(z_0, z_1) \longmapsto \frac{z_0}{z_1} \qquad S^3 \subseteq \mathbb{C} \times \mathbb{C}$$

and μ is the Hopf map. This is a fibre bundle with fibre S^1. (In general it is common to write $F \longrightarrow E \xrightarrow{p} B$ for fibrations) We have a les

$$
\begin{array}{ccccccccccccccc}
\pi_3 S^1 & \longrightarrow & \pi_3 S^3 & \longrightarrow & \pi_3 S^2 & \xrightarrow{\partial} & \pi_2 S^1 & \longrightarrow & \pi_2 S^3 & \longrightarrow & \pi_2 S^2 & \xrightarrow{\partial} & \pi_1 S^1 & \longrightarrow & 0 \\
\| & & \wr\| & & \| & & \| & & \| & & \wr\| & & \wr\| & & \\
0 & & \mathbb{Z} & & ? & & 0 & & 0 & & \mathbb{Z} & & \mathbb{Z} & & \\
 & & \text{(Hurewicz)} & & & & & & & & & & & &
\end{array}
$$

$\Rightarrow$ $\boxed{\pi_3 S^2 \cong \mathbb{Z}}$. Moreover we have $\boxed{\pi_i S^3 \cong \pi_i S^2}$ $\forall i \geq 3$

since $\pi_{i-1} S^1 = \pi_i S^1 = 0$ $\forall i \geq 3$.

As $\mathbb{C}P^1 \cong S^2$, one can generalise this as $S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$

Corollary: $\forall i \geq 3$: $\boxed{\pi_i S^{2n+1} \cong \pi_i \mathbb{C}P^n}$

This fails for $i = 2$: $\pi_2 S^3 = 0$ but $\pi_2 \mathbb{C}P^1 \cong \pi_2 S^2 \cong \mathbb{Z}$.

Taking the limit, one has a fibre bundle $S^1 \longrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$.

$S^\infty \simeq *$, $0 \longrightarrow \pi_2 \mathbb{C}P^\infty \xrightarrow{\partial} \pi_1 S^1 \longrightarrow 0$ $\Rightarrow$ $\boxed{\pi_i \mathbb{C}P^\infty = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & \text{else} \end{cases}}$

$\Rightarrow$ $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ Eilenberg–MacLane space

Another way: $\pi_i \mathbb{C}P^\infty \cong \pi_i \mathbb{C}P^n$ for $n \gg 0$.

Similarly: $\boxed{\pi_2 \mathbb{C}P^n \cong \mathbb{Z}}$ $\forall n \geq 1$. This also follows from Hurewicz.

If $E \xrightarrow{p} B$ is a covering space, B path-connected then $E \xrightarrow{p} B$ is a fibre bundle with a discrete fibre.

$$\Rightarrow \pi_i E \xrightarrow{\cong} \pi_i B \quad \forall i \geq 2, \quad \boxed{\pi_i S^n \cong \pi_i \mathbb{R}P^n} \quad \forall i \geq 2$$

$$\Rightarrow \boxed{\pi_i \mathbb{R}P^\infty = \begin{cases} \mathbb{Z}/2 & i=0 \\ 0 & \text{else} \end{cases}} \qquad \mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$$

<u>Example</u>. G a Lie group, $H \leq G$ a closed subgroup ($\Rightarrow$ H is a Lie subgroup).

$G \xrightarrow{p} G/H$ is a fibre bundle w/ fibre H. (PO)

G/H is the so-called homogeneous space

One can show that p is a submersion, i.e. the derivative is surjective.

Submersion thm. $\Rightarrow$ p has a local section at every point:

$\forall x \in G/H$, $x \in U$, $s: U \longrightarrow G$ section of p

$$\begin{aligned} H \times U &\longrightarrow p^{-1}(U) \\ (h, x) &\longmapsto h \cdot s(x) \end{aligned}$$

$$SO(n-1) \hookrightarrow SO(n) \twoheadrightarrow SO(n)/SO(n-1) \cong S^{n-1}$$
$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$
is a fibre bundle (PO)

$\Rightarrow$ $\boxed{\pi_i SO(n-1) \cong \pi_i SO(n)}$ $\forall i < n-1$ by les (use $\pi_i S^{n-1} = 0$)

$\Rightarrow$ for $n \gg 0$: $\pi_i SO(n)$ stabilises and $\pi_i SO(n) \cong \pi_i SO$

where <u>SO</u> $:= \bigcup_{n \geq 0} SO(n)$ (Bott also computes these)

Similarly: $U(n-1) \hookrightarrow U(n) \twoheadrightarrow U(n)/U(n-1)$

$\Rightarrow$ $\pi_i U(n)$ is independent of n for $n \gg 0$

<u>U</u> $:= \bigcup_{n \geq 0} U(n)$ $\Rightarrow$ $\pi_i U \cong \underset{n \gg 0}{\pi_i U(n)} \cong \begin{cases} \mathbb{Z} & 2 \nmid i \\ 0 & 2 \mid i \end{cases}$ (Bott)

Note that $\pi_i S^n$ also stabilises in a certain range:

$$\pi_{k+n} S^n \quad \text{if} \quad k < n$$